Nonlinear superposition formula for $N = 1$ supersymmetric KdV Equation

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Abstract

In this paper, we derive a Bäcklund transformation for the supersymmetric Korteweg-de Vries equation. We also construct a nonlinear superposition formula, which allows us to rebuild systematically for the supersymmetric KdV equation the soliton solutions of Carstea, Ramani and Grammaticos.

The celebrated Korteweg-de Vries (KdV) equation was extended into super framework by Kupershmidt [3] in 1984. Shortly afterwards, Manin and Radul [7] proposed another super KdV system which is a particular reduction of their general supersymmetric Kadomstev-Petviashvili hierarchy. In [8], Mathieu pointed out that the super version of Manin and Radul for the KdV equation is indeed invariant under a space supersymmetric transformation, while Kupershmidt's version does not. Thus, the Manin-Radul's super KdV is referred to the supersymmetric KdV equation.

We notice that the supersymmetric KdV equation has been studied extensively in literature and a number of interesting properties has been established. We mention here the infinite conservation laws [8], bi-Hamiltonian structures [10], bilinear form [9][2], Darboux transformation [6].

By the constructed Darboux transformation, Mañas and one of us calculated the soliton solutions for the supersymmetric KdV system. This sort of solutions was also obtained by Carstea in the framework of bilinear formalism [1]. However, these solutions are characterized by the presentation of some constraint on soliton parameters. Recently, using super-bilinear operators, Carstea, Ramani and Grammaticos [2] constructed explicitly new two- and three-solitons for the supersymmetric KdV equation. These soliton solutions are interesting since they are free of any constraint on soliton parameters. Furthermore, the fermionic part of these solutions is dressed through the interactions.

In addition to the bilinear form approach, Bäcklund transformation (BT) is also a powerful method to construct solutions. Therefore, it is interesting to see if the soliton solutions of Carstea-Ramani-Grammaticos can be constructed by BT approach. In this paper, we first construct a BT for the supersymmetric KdV equation. Then, we derive a nonlinear superposition formula. In this way, the soliton solutions can be produced

systematically. We explicitly show that the two-soliton solution of Carstea, Ramani and Grammaticos appears naturally in the framework of BT.

To introduce the supersymmetric extension for the KdV equation, we recall some terminology and notations. The classical spacetime is (x, t) and we extend it to a super-spacetime (x, t, θ), where θ is a Grassmann odd variable. The dependent variable $u(x, t)$ in the KdV equation is replaced by a fermionic variable $\Phi = \Phi(x, t, \theta)$. Now the super-symmetric KdV equation reads as

$$\Phi_t - 3(\Phi\mathcal{D}\Phi)_x + \Phi_{xxx} = 0, \tag{1}$$

where $\mathcal{D} = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial x}$ is the superderivative. Mathieu found the following supersymmetric version of Gardner type map

$$\Phi = \chi + \epsilon\chi_x + \epsilon^2\chi(\mathcal{D}\chi), \tag{2}$$

where ϵ is an ordinary (bosonic) parameter.

It is easy to show that χ satisfies the following supersymmetric Gardner equation

$$\chi_t - 3(\chi\mathcal{D}\chi)_x - 3\epsilon^2(\mathcal{D}\chi)(\chi\mathcal{D}\chi)_x + \chi_{xxx} = 0. \tag{3}$$

This map was used in [8] to prove that there exists an infinite number of conservation laws for the supersymmetric KdV equation (1). In the classical case, Gardner type of map was studied extensively by Kupershmidt [4]. It is well known that such map may be used to construct interesting BT. We will show that it is also the case for the supersymmetric KdV equation.

We notice that the supersymmetric Gardner equation (3) is invariant under $\epsilon \to -\epsilon$. The new solution of the supersymmetric KdV equation corresponding to with $-\epsilon$ is denoted as $\tilde{\Phi}$. Thus we have

$$\tilde{\Phi} = \chi - \epsilon\chi_x + \epsilon^2\chi(\mathcal{D}\chi). \tag{4}$$

From above relations (3-4), we find

$$\begin{aligned} \Phi - \tilde{\Phi} &= 2\epsilon\chi_x, & (5)\\ \Phi + \tilde{\Phi} &= 2\chi + 2\epsilon\chi(\mathcal{D}\chi). & (6) \end{aligned}$$

Let us introduce the potentials as follows

$$\Phi = \Psi_x, \quad \tilde{\Phi} = \tilde{\Psi}_x,$$

thus, the equation (5) provides us

$$\chi = \frac{1}{2\epsilon}(\Psi - \tilde{\Psi}), \tag{7}$$

Eliminating χ between the equations (6) and (7), we arrive at a BT

$$(\Psi + \tilde{\Psi})_x = \lambda(\Psi - \tilde{\Psi}) + \frac{1}{2}(\Psi - \tilde{\Psi})(\mathcal{D}\Psi - \mathcal{D}\tilde{\Psi}), \tag{8}$$

where $\lambda = 1/\epsilon$ is the Bäcklund parameter.

The transformation (8) is in fact the spatial part of BT. Its temporal counterpart can be easily worked out. We also remark here that the BT is reduced to the well known BT for the classical KdV equation if $\eta = 0$, as it should be.

A BT can be used to generate special solutions. If we start with the trivial solution $\tilde{\Psi} = 0$, we obtain

$$\Psi = -\frac{2a(\zeta + \theta\lambda)e^{\lambda x - \lambda^3 t}}{1 + ae^{\lambda x - \lambda^3 t}}, \tag{9}$$

where a is an Grassmann even constant and ζ Grassmann odd one. When $a > 0$, we have 1-soliton solution for the supersymmetric KdV equation (1). For $a < 0$, we have a singular solution. (cf. [11] for the classical KdV equation)

Next, we shall work out a superposition formula. To this end, we start with the seed solution Ψ_0. After one step transformation, our seed is transformed to Ψ_1 with parameter λ_1, to Ψ_2 with parameter λ_2, respectively. Then we do second step transformation: starting with Ψ_1, we obtain Ψ_{12} with λ_2 while starting with Ψ_2, we obtain Ψ_{21} with λ_1. By Bianchi's theorem of permutability, one should have $\Psi_{12} = \Psi_{21}$. For convenience, we denote

$$\Psi_3 =: \Psi_{12} = \Psi_{21}.$$

$$\begin{tikzcd}
& \Psi_1 \arrow[dr, "\lambda_2"] & \\
\Psi_0 \arrow[ur, "\lambda_1"] \arrow[dr, "\lambda_2"'] & & \Psi_3 \\
& \Psi_2 \arrow[ur, "\lambda_1"'] &
\end{tikzcd}$$

Bianchi's diagram

We list down the relations obtained

$$(\Psi_0 + \Psi_1)_x = \lambda_1(\Psi_1 - \Psi_0) + \frac{1}{2}(\Psi_1 - \Psi_0)(\mathcal{D}\Psi_1 - \mathcal{D}\Psi_0), \tag{10}$$

and

$$(\Psi_0 + \Psi_2)_x = \lambda_2(\Psi_2 - \Psi_0) + \frac{1}{2}(\Psi_2 - \Psi_0)(\mathcal{D}\Psi_2 - \mathcal{D}\Psi_0), \tag{11}$$

and

$$(\Psi_1 + \Psi_3)_x = \lambda_2(\Psi_3 - \Psi_1) + \frac{1}{2}(\Psi_3 - \Psi_1)(\mathcal{D}\Psi_3 - \mathcal{D}\Psi_1), \tag{12}$$

and

$$(\Psi_2 + \Psi_3)_x = \lambda_1(\Psi_3 - \Psi_2) + \frac{1}{2}(\Psi_3 - \Psi_2)(\mathcal{D}\Psi_3 - \mathcal{D}\Psi_2). \tag{13}$$

Subtraction (10) form (11), we have

$$\begin{aligned}(\Psi_2-\Psi_1)_x &= \lambda_2\Psi_2-\lambda_1\Psi_1+(\lambda_1-\lambda_2)\Psi_0+\frac{1}{2}\Psi_2(\mathcal{D}\Psi_2)-\frac{1}{2}\Psi_2(\mathcal{D}\Psi_0)\\ &\quad -\frac{1}{2}\Psi_0(\mathcal{D}\Psi_2)-\frac{1}{2}\Psi_1(\mathcal{D}\Psi_1)+\frac{1}{2}\Psi_1(\mathcal{D}\Psi_0)+\frac{1}{2}\Psi_0(\mathcal{D}\Psi_1),\end{aligned} \tag{14}$$

similarly, from (12) and (13) we have

$$\begin{aligned}(\Psi_2-\Psi_1)_x &= \left[\lambda_1-\lambda_2+\frac{1}{2}(\mathcal{D}\Psi_1)-\frac{1}{2}(\mathcal{D}\Psi_2)\right]\Psi_3+\frac{1}{2}(\Psi_1-\Psi_2)(\mathcal{D}\Psi_3)\\ &\quad +\lambda_2\Psi_1-\lambda_1\Psi_2+\frac{1}{2}\Psi_2(\mathcal{D}\Psi_2)-\frac{1}{2}\Psi_1(\mathcal{D}\Psi_1).\end{aligned} \tag{15}$$

Now equations (14) and (15) give us

$$\begin{aligned}&\left[\lambda_1-\lambda_2+\frac{1}{2}(\mathcal{D}\Psi_1)-\frac{1}{2}(\mathcal{D}\Psi_2)\right](\Psi_3-\Psi_0)\\ &\quad +\frac{1}{2}(\Psi_1-\Psi_2)\left[(\mathcal{D}\Psi_3)+2\lambda_1+2\lambda_2-(\mathcal{D}\Psi_0)\right]=0.\end{aligned} \tag{16}$$

The equation (16) is a differential equation for Ψ_3. Solving it we obtain

$$\Psi_3=\Psi_0-\frac{(\lambda_1+\lambda_2)(\Psi_1-\Psi_2)}{\lambda_1-\lambda_2+(\mathcal{D}\Psi_1)-(\mathcal{D}\Psi_2)}, \tag{17}$$

this is the superposition formula for supersymmetric KdV equation (1). Substituting $\Psi_i=\eta_i+\theta v_i$ $(i=0,1,2,3)$ into (17), our superposition formula may be decomposed into

$$\begin{aligned}v_3 &= v_0-\frac{(\lambda_1+\lambda_2)(v_1-v_2)}{\lambda_1-\lambda_2+v_1-v_2}-\frac{(\lambda_1+\lambda_2)(\eta_1-\eta_2)(\eta_{1,x}-\eta_{2,x})}{(\lambda_1-\lambda_2+v_1-v_2)^2},\\ \eta_3 &= \eta_0-\frac{(\lambda_1+\lambda_2)(\eta_1-\eta_2)}{(\lambda_1-\lambda_2+v_1-v_2)}.\end{aligned}$$

It is clear that our nonlinear superposition formula reduces to the well-known superposition formula for the KdV as it should be. The advantage to have a superposition formula is that it is an algebraic one and can be used easily to find solutions.

Using the solutions (9) as our seeds, we may construct a 2-soliton solution of (1) by means of our superposition formula. Indeed, let

$$\Psi_0=0,\quad \Psi_1=-\frac{2a_1(\zeta_1+\theta\lambda_1)e^{\lambda_1x-\lambda_1^3t}}{1+a_1e^{\lambda_1x-\lambda_1^3t}},\quad \Psi_2=-\frac{2a_2(\zeta_2+\theta\lambda_2)e^{\lambda_2x-\lambda_2^3t}}{1+a_2e^{\lambda_2x-\lambda_2^3t}}$$

then from our nonlinear superposition formula (17), we obtain

$$\Psi_3=\frac{2\left[\zeta_1a_1e^{\delta_1}-\zeta_2a_2e^{\delta_2}+(\zeta_1-\zeta_2)a_1a_2e^{\delta_1+\delta_2}+\theta(\lambda_1a_1e^{\delta_1}-\lambda_2a_2e^{\delta_2}+(\lambda_1-\lambda_2)a_1a_2e^{\delta_1+\delta_2})\right]}{(\lambda_1+\lambda_2)^{-1}\left[(\lambda_1-\lambda_2)-(\lambda_1+\lambda_2)a_1e^{\delta_1}+(\lambda_1+\lambda_2)a_2e^{\delta_2}-(\lambda_1-\lambda_2)a_1a_2e^{\delta_1+\delta_2}\right]}$$

where $\delta_i = \lambda_i x - \lambda_i^3 t + \theta\zeta_i$ $(i = 1, 2)$. Now, taking

$$\lambda_1 > \lambda_2, \quad a_1 = -\frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2}, \qquad a_2 = \frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2}$$

we recover the 2-soliton solution found first by Carstea, Ramani and Grammaticos [2]. As in the classical KdV case [11], we generate this 2-soliton from a regular solution and a singular solution.

We could continue this process to build the higher soliton solutions and the calculation will be tedious but straightforward.

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